Algebraic constraints on Tau functions of P reduced KP with String equation

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Abstract

Give a convenient new approach to obtain the algebraic constraints on tau functions of p-reduced KP with the constraints of string equation. The classical results about algebraic constraints on associated tau function are included.

Keywords: W_p^+ constrain, ASvM, Virasoro constrain, String equation

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§1. Introduction

Ed.Witten and Kontsevich[1, 2, 3] gave a classical result that solutions of KdV equation with the constraint of string equation corresponds to partition function of 2D quantum gravity or generating function of intersection number. And Kontsevich[1] showed the associative solution by matrix integral. Since the solution of the KdV equation can be characterized by a single function of tau function[12], it also showed that the tau function is a vacuum vector for the Virasoro Algebra. Since integrable system has tightly connection with string theory and intersection theory[4, 5], the results can be show in language of integrable system[6, 7, 8]. Also it was considered to extend the results from 2-reduced KP to p-reduced KP hierarchy, and conjecture[9, 10] that for general p the tau function which satisfy the string equation and p-reduced KP hierarchy are equivalent to the vacuum vector of a W_p-algebra. Goeree[11] shows that it is true when $p = 3$. The case for higher p was also researched[10, 4, 8]. And extend the corresponding results to other integrable hierarchy is also an interesting topic, such as for BKP, qKP, mKP, etc.

In the paper we show a new convenient approach for obtain the algebra constraints on tau functions of p-reduced KP constrained by string equation. With this method we can naturally get the condition of $t_{kp} = 0$. And these algebraic constrains include the classic results about Virasora constraints and W_p^+ constrains which is the conjecture[9]. The whole process of computation could greatly simplify by the tool of the ASvM formula[7, 14, 15]. In these process we found that we can not directly let $t_{kp} = 0$, which makes $t_{kp} = 0$ as an additional condition. And we also found for obtain strictly Virasoro, we should add a constant on the item of constraint of $\mathcal{L}_0$, not let all constant c=0. Fortunately we found the additional constant just cancel with the item including t_{kp} in the next higher order algebraic constraint of $k = -1$. Inspire by that we extend these method to the whole algebraic constraints and obtain our results of theorem 3.2. And because the additional constant do not destroy the algebra structure and we could stop the process at a fixed number, we also obtain Theorem 3.3.

The organization of the paper is as follows. In section 2, for self-contained we give a brief description of KP hierarchy. In section 3, we deduced the algebraic constraints on tau function.

§2. KP hierarchy

To be self-contained, we list some properties of KP hierarchy, based on detailed research in [13].

$$f(t) = f(x + t_1, t_2, \cdots, t_j, \cdots),$$

for any$j \in \mathbb{Z}$

$$\partial^j \circ f = \sum_{i=0}^{\infty} \binom{j}{i} (\partial^i f)\partial^{j-i}, \quad \binom{j}{i} = \frac{j(j-1)\cdots(j-i+1)}{i!}. \tag{2.1}$$

Define the KP operator as follow

$$L = \partial + \sum_{j=1}^{\infty} f_j(t)\partial^{-j}. \tag{2.2}$$

Denote by $L_+ = \sum_{j=0}^{d} f_j(n)\partial^j$, and by $L_- = \sum_{j=-\infty}^{-1} f_j(n)\partial^j$.

The KP-hierarchy [13] is a family of evolution equation in infinitely many variables $t = (x + t_1, t_2, \cdots)$

$$\frac{\partial L}{\partial t_i} = [(L^i)_+, L]. \tag{2.3}$$

There are dressing operator $\Phi(t)$

$$\Phi(t) = 1 + \sum_{j=1}^{\infty} w_j(t)\partial^{-j},$$

and

$$L = \Phi(t) \circ \triangle \circ \Phi(t)^{-1}. \tag{2.4}$$

There are wave function $w(t,z)$ and adjoint wave function $w^*(t,z)$,

$$w(t,z) = \Phi(t) exp(\sum_{i=1}^{\infty} t_i z^i) \tag{2.5}$$

and

$$w^*(t,z) = (\Phi^{-1}(t))^* exp(\sum_{i=1}^{\infty} -t_i z^i). \tag{2.6}$$

And a tau function $\tau(t)$ for KP exists, which satisfies that

$$w(t,z) = \frac{\tau(t - [z^{-1}])}{\tau(t)} exp(\sum_{i=1}^{\infty} t_i z^i) \tag{2.7}$$

and

$$w^*(t,z) = \frac{\tau(t + [z^{-1}])}{\tau(t)} exp(\sum_{i=1}^{\infty} -t_i z^i), \tag{2.8}$$

where denote $[z] = (z, \frac{z^2}{2}, \frac{z^3}{3}, \cdots)$. Also introduce the $G(z)$ operator, which action is $G_t(z)f(t) = f(t - [z])$ and $G^*_{t'}(z)f(t') = f(t' + [z])$.

There are vertex operators

$$X(\lambda, \mu) =: \exp \sum_{-\infty}^{\infty} (\frac{P_i}{i\lambda^i} - \frac{P_i}{i\mu^i}) : \tag{2.9}$$

where where

$$P_i = \begin{cases} \partial_i, & i > 0 \\ \\ |i| t_{|i|}, & i \leq 0 \end{cases}$$

And there are additional symmetries. Define

$$\Gamma = \sum_{i=1}^{\infty} i t_i \partial^{i-1}, \tag{2.10}$$

and

$$M = \Phi(t) \circ \Gamma \circ \Phi(t)^{-1}. \tag{2.11}$$

then the additional symmetries are defined by

$$\partial^*_{ml} \Phi(t) = -(M^m L^l)_- \circ \Phi(t). \tag{2.12}$$

Here for convenient the symbols have a slight different from [13].

Talyor expand the $X(\lambda, \mu)$ in μ at the point of λ, there are the following

$$X(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{n=-\infty}^{\infty} \lambda^{-n-m} W_n^{(m)}. \tag{2.13}$$

Where

$$\sum_{n=-\infty}^{\infty} \lambda^{-n-m} W_n^{(m)} = \partial_\mu^m|_{\mu=\lambda} X(\lambda, \mu).$$

and the first items are

$$\begin{aligned}
W_n^{(0)} &= \delta_{n,0}, \\
W_n^{(1)} &= P_n, \\
W_n^{(2)} &= \sum_{i+j=n} : P_i P_j : -(n+1) P_n, \\
W_n^{(3)} &= \sum_{i+j+k=n} : P_i P_j P_k : -\frac{3}{2}(n+2) \sum_{i+j=n} : P_i P_j : +(n+1)(n+2) P_n, \\
W_n^{(4)} &= P_n^{(4)} - 2(n+3) P_n^{(3)} + (2n^2 + 9n + 11) P_n^{(2)} - (n+1)(n+2)(n+3) P_n \qquad (2.14) \\
&\cdots\cdots \qquad (2.15)
\end{aligned}$$

Here denote

$$\begin{aligned}
P_n^{(2)} &= \sum_{i+j=n} : P_i P_j : \qquad (2.16) \\
P_n^{(3)} &= \sum_{i+j+k=n} : P_i P_j P_k : \qquad (2.17) \\
P_n^{(4)} &= \sum_{i+j+k+l=n} : P_i P_j P_k P_l : - \sum_{i+j=n} : ij P_i P_j : \qquad (2.18) \\
&\cdots\cdots \qquad (2.19)
\end{aligned}$$

And there are ASvM formula for KP hierarchy[7, 14, 15], that is

$$\partial^*_{m,l+m} \tau(n; t) = \frac{W_l^{(m+1)} \cdot \tau(t)}{m+1}, \tag{2.20}$$

which hold for $m \geq 0$ and for all l.

§3 Algebraic constraints on tau functions

In this section we deduce the algebraic constraints on tau functions which satisfy the p-reduced KP and sting equation.

First we should oobtain the action of additional symmetries on tau functions from the action of additional symmetries on wave functions. That is the following lemma

lemma3.1 For $m \geq 0$, and $\forall l$, there are

$$\partial^*_{m,m+l} w(t,z) = (G(z)-1)\frac{W_l^{(m+1)}/m+1 \cdot \tau(t)}{\tau(t)} \cdot w(t,z). \tag{3.1}$$

Proof: There are

$$\begin{aligned} &\partial^*_{m,m+l} w(t,z) \\ =&\partial^*_{m,m+l}\left(\frac{G(z)\tau(t)}{\tau(t)} \exp\sum_{i=1}^{\infty}(t_i z^i)\right) \\ =&\frac{\tau(t)\cdot G(z)(\partial^*_{m,m+l}\tau(t)) - G(z)\tau(t)\cdot \partial^*_{m,m+l}\tau(t)}{\tau^2(t)} \exp\sum_{i=1}^{\infty}(t_i z^i) \\ =&\left((G(z)-1)\frac{W_l^{(m+1)}/m+1\cdot\tau(t)}{\tau(t)}\right)w(t,z). \end{aligned} \tag{3.2}$$

□

Now consider the p-reduced KP and string equation. The string equation [13] means that find two differential operators satisfy $[P,Q]=1$, that is

$$[L^p, \frac{1}{p}(ML^{-p+1})_+] = 1. \tag{3.3}$$

which is equivalent the following two conditions. First the p-reduced condition

$$(L^{kp})_- = 0, \tag{3.4}$$

which means L is independent on t_{kp}. And that also means $\tau(t)$ is independent on t_{kp}.

Second satisfy the following identities

$$\partial^*_{1,-p+1} L = 0. \tag{3.5}$$

We found that the condition of (3.5) could lead to the different values of the dressing operator $\Phi(t)$, that is these different values of dressing operator can all keep the condition of (3.5). And the different values lead to different algebraic constraints on tau functions. For example, if take

$$\partial^*_{1,-p+1}\Phi(t) = 0, \tag{3.6}$$

it can satisfy the condition (3.5) and obtain

$$\partial^*_{1,-p+1} w(t,z) = 0. \tag{3.7}$$

And from the lemma 3.1, it obtain

$$\frac{W^{(2)}_{-p}\cdot\tau(t)}{2}=c\tau(t). \tag{3.8}$$

that is

$$(\frac{1}{2}\sum_{i+j=-p}:P_iP_j:-(-p+1)P_{-p})\cdot\tau(t)=c\tau(t). \tag{3.9}$$

but because the left hand $\frac{W^{(2)}_{-p}\cdot\tau(t)}{2}$ include the variable t_p, but the right hand $\tau(t)$ does not it lead to that no solution of the $\tau(t)$ functions satisfy the above identity.

So to keep the existence of the solution ,we let

$$\partial^*_{1,-p+1}\Phi(t)=-\frac{p-1}{2}L^{-p}, \tag{3.10}$$

it also keep the equation (3.5), but it obtain

$$\partial^*_{1,-p+1}w(t,z)=-\frac{p-1}{2}z^{-p}\cdot w(t,z). \tag{3.11}$$

Use lemma 3.1, that is

$$(G(z)-1)\frac{W^{(2)}_{-p}/2\cdot\tau(t)}{\tau(t)}=-\frac{p-1}{2}z^{-p}. \tag{3.12}$$

which means

$$(G(z)-1)\frac{\mathcal{L}_{-p}\cdot\tau(t)}{\tau(t)}=0. \tag{3.13}$$

That is

$$\frac{1}{2}\sum_{i+j=-p}:P_iP_j:\tau(t)=\mathcal{L}_{-p}\tau(t)=c\tau(t), \tag{3.14}$$

then obtain the same results in [6].

Note that the identity of (3.10) is very important, and we can obtain the whole algebraic constraints only from it and the p-reduced condition. So (3.14) is also the basic equation that tau function should satisfy.

With the definition of additional symmetries the condition (3.10) is equivalent to

$$(ML^{-p+1})_-=\frac{p-1}{2}L^{-p}. \tag{3.15}$$

And only start with the (3.15), together with the p-reduced conditions, there are a general results[8]

$$\begin{aligned}(M^jL^{kp+j})_- &= \prod_{r=0}^{j-1}(\frac{p-1}{2}-r)L^{-p} && k=-1 \quad j=1,2,\cdots && (3.16)\\ &=0 && k=0,1,2,\cdots \quad j=1,2,\cdots && (3.17)\end{aligned}$$

In term of dressing operators that is

$$\partial^*_{j,kp+j}\Phi(t) = -\prod_{r=0}^{j-1}(\frac{p-1}{2}-r)L^{-p}\circ\Phi(t) \qquad k=-1 \quad j=1,2,\cdots \tag{3.18}$$

$$= 0 \qquad k=0,1,2,\cdots \quad j=1,2,\cdots \tag{3.19}$$

And In term of the wave functions that is

$$\partial^*_{j,kp+j}w(t,z) = -\prod_{r=0}^{j-1}(\frac{p-1}{2}-r)z^{-p} \qquad k=-1 \quad j=1,2,\cdots \tag{3.20}$$

$$= 0 \qquad k=0,1,2,\cdots \quad j=1,2,\cdots \tag{3.21}$$

And with the above methods of ASvM, there are

$$\begin{aligned}(G(z)-1)\frac{W^{(j+1)}_{kp}/j+1\cdot\tau(t)}{\tau(t)} &= -\prod_{r=0}^{j-1}(\frac{p-1}{2}-r)z^{-p} \qquad k=-1 \quad j=1,2,\cdots \\ &= 0 \qquad k=0,1,2,\cdots \quad j=1,2,\cdots \end{aligned}\tag{3.22}$$

These are the equations which the tau functions under the constraints of p-reduced conditions and string equation should satisfy. And these equations are equivalent to (3.10) and the reduce conditions. And we could deduce the algebraic constraints on tau function only from the above equations. Notice that the item of $k=-1$ play an important role in above equation.

Notice that the above equation lead to a lot of uncertain constant, and the differen value of these constants may lead to different algebraic structure. Here we show a method to obtain a algebraic structure including the classical results.

Start with $k=-1$, $j=1$, that is the result (3.14). In all we obtain

$$\mathcal{L}_{-p}\cdot\tau(t) = c\tau(t) \qquad k=-1 \quad j=1 \tag{3.23}$$

$$\frac{W^{(2)}_{kp}}{2}\cdot\tau(t) = c\tau(t) \qquad k=0,1,2,\cdots \quad j=1 \tag{3.24}$$

And noticed that when $k\geq 0$, the $P_{kp}=\partial_{kp}$ or $P_{kp}=0$. And because the p-reduced conditions the $\tau(t)$ is independent on t_{kp}, there are

$$P_{kp}\cdot\tau(t) = 0 \quad k=0,1,\cdots \tag{3.25}$$

So in (3.24) we can get rid of the items of P_{kp} in $W^{(2)}_{kp}$. Then the remain of (3.24) are $\mathcal{L}_{kp}$. So in all it obtains

$$\mathcal{L}_{kp}\cdot\tau(t) = c\tau(t) \quad k=-1,0,1,2,\cdots \tag{3.26}$$

Here c is a constant. With the expression of $\mathcal{L}$ notice that under the condition of $\partial_{kp}\tau(t)=0$ the $\mathcal{L}_{kp}$ can naturally get rid of the items which include the variable t_{kp}. So we need not to let $t_{kp}=0$ in special or as some additional conditions. It was satisfied naturally. For convenient also denote $\mathcal{L}|_{t_{kp}=0}$ by $\mathcal{L}$.

If we want to get the Virasoro constraints, we can not simply let all $c = 0$ because the $\mathcal{L}_{kp}$ do not form the virasoro. The Virasoro satisfy the commutatiom

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m} + \frac{1}{12}(n^3 - n)\delta_{n+m}, \tag{3.27}$$

but the $\mathcal{L}_{kp}$ satisfy

$$[\mathcal{L}_{np}, \mathcal{L}_{mp}] = (np-mp)\mathcal{L}_{np+mp} + \frac{1}{12}((np)^3 - np)\delta_{np+mp}. \tag{3.28}$$

So to get the Virasoro constrains we should firstly let

$$L_n = \frac{1}{p}\mathcal{L}_{np} \quad n = -1, 0, 1, 2, \cdots \tag{3.29}$$

but it does not enough. Secondly to keep the action of $\mathcal{L}_0$ on tau function is correct, that is to keep

$$[\mathcal{L}_{-p}, \mathcal{L}_p]\tau(t) = 0, \tag{3.30}$$

we should let

$$\mathcal{L}_0\tau(t) = -\frac{1}{24}(p^2 - 1)\tau(t), \tag{3.31}$$

which means for $\mathcal{L}_0$ we take constant $c = -\frac{1}{24}(p^2-1)$ and for other $k = -1, 1, 2, \cdots$, let $c = 0$ which is

$$\mathcal{L}_{kp}\tau(t) = 0 \quad k = -1, 1, 2, \cdots \tag{3.32}$$

The change of (3.31) is very important in higher order algebraic constraints and we shall show it later. Then take

$$L_0 = \frac{1}{p}\mathcal{L}_0 + \frac{1}{24p}(p^2 - 1), \tag{3.33}$$

and because the constant do not effect the Poisson Bracket there are correct Virasoro relation

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}(n^3 - n)\delta_{n+m}, \tag{3.34}$$

and

$$L_n\tau = 0 \quad n = -1, 0, 1, \cdots \tag{3.35}$$

So we get the Virasoro constraints as a subalgebra. For $p = 2$, that is the classical results[1, 2, 3].

Now go on this process. consider $j = 2$ in equations (3.22) which lead to

$$\begin{aligned}
&(G(z)-1)\frac{\frac{1}{3}(P^{(3)}_{kp} - \frac{3}{2}(kp+2)P^{(2)}_{kp} + (kp+1)(kp+2)P_{kp}) \cdot \tau(t)}{\tau(t)} \\
&= -\frac{p-3}{2} \cdot \frac{p-1}{2} z^{-p} \quad k = -1 \quad j = 1, 2, \cdots \\
&= 0 \qquad\qquad k = 0, 1, 2, \cdots \quad j = 1, 2, \cdots
\end{aligned} \tag{3.36}$$

First consider $k=-1$, with the results of $j=1$, there are

$$(G(z)-1)\frac{\frac{1}{3}(P_{-p}^{(3)}+(p-1)(p-2)P_{-p})\cdot\tau(t)}{\tau(t)}=-\frac{p-3}{2}\cdot\frac{p-1}{2}z^{-p} \tag{3.37}$$

Also same to $j=1$, to keep the existence of the solution of tau function, there should get rid of the items which include the variables t_{kp}. But we can not let $t_{kp}=0$ directly which is as a additional conditions. we should get rid of them naturally. Notice that in $P_{-p}^{(3)}$, the items include t_{kp} is $3\cdot P_{-kp}\cdot\sum_{i+j=kp-p}P_iP_j$. Use the results of $j=1$, we only need consider $3\cdot P_{-p}\cdot\sum_{i+j=0}P_iP_j$. So consider the the change of (3.31), the equation (3.37) jus show that

$$(P_{-p}^{(3)}|_{\text{without}\quad t_{kp})}\cdot\tau(t)=c\tau(t), \tag{3.38}$$

which means the the change of (3.31) just cancels with the items include t_p. So let $c=0$ there are

$$P_{-p}^{(3)}|_{t_{kp}=0}\cdot\tau(t)=0. \tag{3.39}$$

As for the equations of $k>0$, we can also naturally get rid of the items which include the t_{kp} by the results of $j=1$. Also let the corresponding $c=0$, there are

$$P_{kp}^{(3)}|_{t_{kp}=0}\cdot\tau(t)=0\quad k=1,2,\cdots \tag{3.40}$$

As for the item of $k=0$, the equation is

$$(G(z)-1)\frac{\frac{1}{3}(P_0^{(3)}-3P_0^{(2)})\cdot\tau(t)}{\tau(t)}=0, \tag{3.41}$$

for satisfy the equation and by the same analysis, there is

$$P_0^{(3)}|_{t_{kp}=0}\cdot\tau(t)=c\tau(t). \tag{3.42}$$

Now we should fix on the constant c. If we want to obtain the next higher order algebraic constraints, we do not give c an arbitrary value. If we stop the process here, just let $c=0$. Otherwise notice that the change of (3.31) and inspire by that. We found that if we give c a appropriate value the higher the equation $j=3,k=-1$ could holds, which also means that we can obtain the value of c from the next higher algebraic constraint equation of $k=-1$. Here for $j=2,k=0$, $c=0$, we shall show it in the following.

Continue the process, consider the case of $j=3$,

$$\begin{aligned}&(G(z)-1)\frac{1}{\tau(t)}\cdot\\&\frac{1}{4}(P_{kp}^{(4)}-2(kp+3)P_{kp}^{(3)}+(2(kp)^2+9(kp)+11)P_{kp}^{(2)}-(kp+1)(kp+2)(kp+3)P_{kp})\cdot\tau(t)\\&=-\frac{p-5}{2}\cdot\frac{p-3}{2}\cdot\frac{p-1}{2}z^{-p}\quad k=-1\quad j=1,2,\cdots\\&=0\qquad\qquad k=0,1,2,\cdots\quad j=1,2,\cdots\end{aligned} \tag{3.43}$$

we can also use the method above and the same analysis and the results of $j = 1, 2$ to obtain

$$P_{kp}^{(4)}|_{t_{kp}=0} \cdot \tau(t) = c\tau(t) \quad k = -1, 0, 1, 2, \cdots \tag{3.44}$$

And here the items include the variables t_{kp} also can naturally be get rid of.

And in the process of obtain the result of

$$P_{-p}^{(4)}|_{t_{kp}=0} \cdot \tau(t) = c\tau(t), \tag{3.45}$$

which comes from the equation

$$\begin{aligned}&(G(z) - 1)\frac{1}{\tau(t)} \cdot \frac{1}{4}(P_{-p}^{(4)} + 2(p-3)P_{-p}^{(3)} + (p-1)(p-2)(p-3)P_{kp}) \cdot \tau(t) \\ &= -\frac{p-5}{2} \cdot \frac{p-3}{2} \cdot \frac{p-1}{2} z^{-p}.\end{aligned} \tag{3.46}$$

Notice that from the change of (3.31),there is

$$\sum_{i+j=0} P_i P_j = -\frac{1}{12}(p^2 - 1). \tag{3.47}$$

For get rid of the items include of t_p, there should have the following simple equation

$$c - \frac{(p^2-1)(p-3)}{8} + \frac{(p-1)(p-3)(p-3)}{4} = \frac{(p-1)(p-3)(p-5)}{8}, \tag{3.48}$$

here c is the constant c for

$$P_0^{(3)}|_{t_{kp}=0} \cdot \tau(t) = c\tau(t). \tag{3.49}$$

That means for satisfying the next higher order algebraic constrains we can obtain the c of $j = 2, k = 0$ by a simple equation (3.48). That is the method how to obtain the value of c of $k = 0$ for all j. The solution of (3.48) is 0, which is the result we mentioned above.

Also we could let $c = 0$, for $k = -1, 1, , 2 \cdots$. And the c for item of $j = 3, k = 0$ could also be obtained from the next higher order algebraic constraints equation by the same method. That is to solve a simple equation similar to (3.48). Also we can stop here and just let the $c = 0$. So we obtain the algebraic constrain of $j = 3$.

And we can continue the process for $j = 4, 5, \cdots$ with similar simple analysis and computation.

Finally there are

Theorem 3.2 If $\tau((t)$ is vacuum vector of the following algebra

$$P_{kp}|t_{kp} = 0 \cdot \tau(t) = 0 \quad k = 0, 1, 2, \cdots \tag{3.50}$$

$$P_{kp}^{(j)}|t_{kp} = 0 \cdot \tau(t) = 0 \quad k = -1, 1, 2, \cdots \quad j = 2, 3, \cdots \tag{3.51}$$

$$(P_0^{(j)} + c_j(p))|t_{kp} = 0 \cdot \tau(t) = 0 \quad j = 2, 3, \cdots \quad c_2(p) = \frac{1}{12}(p^2 - 1), c_3(p) = 0, \cdots \tag{3.52}$$

then $\tau(t)$ is the tau function of p-reduced KP and string equation.

And as j goes bigger the constrains goes more. Because the whole algebra constraints just base on the two basic conditions, p-reduced and (3.10). So to make sure the existence of the solution, we could stop this process at a fixed number, for example at $j = p$. and the existence of the solutions already proved by Kontsevich[1] for $p = 2$, and Adler[8] for general p. And notice that the constant do not destroy the Poisson Bracket and the algebraic structure. So there are

Theorem 3.3 If $\tau((t)$ is vacuum vector of the following W_p^+ algebra

$$P_{kp}|t_{kp} = 0 \cdot \tau(t) = 0 \quad k = 0, 1, 2, \cdots \tag{3.53}$$

$$P_{kp}^{(j)}|t_{kp} = 0 \cdot \tau(t) = 0 \quad k = -1, 1, 2, \cdots \quad j = 2, 3, \cdots, p \tag{3.54}$$

$$(P_0^{(j)} + c_j(p))|t_{kp} = 0 \cdot \tau(t) = 0 \quad j = 2, 3, \cdots, p \quad c_2(p) = \frac{1}{12}(p^2 - 1), c_3(p) = 0, \cdots \tag{3.55}$$

then $\tau(t)$ is the tau function of p-reduced KP and string equation. That is the result of the conjecture[9].

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